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A Spectrum-Generating Algebra for Particles of Spin $\frac{1}{2}$

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Abstract

Spectrum-generating algebra techniques developed in previous works to deal with the problem of one particle in a potential are extended to deal with particles of spin $\frac{1}{2}$. The method is illustrated considering the Pauli and Dirac equations with a Coulomb potential. The way to deal with some other potentials is indicated. In the Pauli equation an intrinsic electric dipole moment term is included. It is particularly remarkable the simplicity with which the solutions are obtained.

1. *Introduction*

The aim of this paper is to extend the algebraic techniques developed mainly by Cordero and Ghirardi (1971a, b, 1972) and by Salamó (1972) from the spectrum-generating algebras (SGA) of rotationally invariant systems to SGA of systems that conserve *total* angular momentum **J**. Formerly the relevant algebra was $SO(2, 1) \times SO(3)_L$ while now the algebra is $SO(2, 1) \times SO(3)_J$ where $\mathbf{J} = \mathbf{L} + \boldsymbol{\sigma}/2$.

The main physically meaningful equations that we are going to consider are the Pauli equation for an electron having an electric dipole moment in a Coulomb potential (Feinberg, 1958; Salpeter, 1958) and the Dirac–Feynman–Gell-Mann equation (Feynman and Gell-Mann, 1958) for the same potential. Cordero et al. (1971a) unsuccessfully attempted to solve the Dirac equation with the same techniques. Barut and Bornzin (1971) have found an algebraic solution for the Dirac equation with an algebra different to ours. As we show, the algebraic method is more general, however, since it allows to solve Pauli-like equations with either a Coulomb potential, or an r^2 potential or even a Morse potential, and all with extra $\boldsymbol{\sigma}$-terms.

The method that we are going to use has been explained by Cordero et al. (1971a, b). Of those results, we need to know that the equation of motion can be written in the form

$$\{p^2 - D(r)\}\psi = 0 \tag{1.1}$$

and we essentially express it in terms of the generators of the SGA as a linear combination of them

$$(1+b)T_0 + (1-b)T_1 - d = 0 \tag{1.2}$$

where the T are generators of $SO(2, 1)$, while b and d are constants.

From here it follows (Cordero et al., 1971a, b) that

$$\frac{1}{2}\frac{h'''}{h'} - \frac{3}{4}\left(\frac{h''}{h'}\right)^2 - Q\left(\frac{h'}{h}\right)^2 - 4b(h')^2 + 2d\frac{h'^2}{h} + \frac{\mathbf{L}^2}{r^2} = D(r) \tag{1.3}$$

The constants b and d are to be determined from self-consistency of equation (1.3), $h(r)$ is a function with arbitrary normalization that we choose to suit (1.3), Q is the Casimir operator of $SO(2, 1)$

$$Q = T_0^2 - T_1^2 - T_2^2 \tag{1.4}$$

and L^2 is the orbital angular momentum operator. The discrete energy spectrum is obtained from tilting (1.2) to pure T_0, giving

$$2\sqrt{b}\left(n + \frac{1}{4} + \sqrt{\frac{1}{4} + q}\right) = d \tag{1.5}$$

where $n = 0, 1, 2, \ldots$ and q is the spectrum of the Casimir operator (1.4). Inside the bracket of equation (1.5) is the discrete spectrum of the compact generator of $SO(2, 1)$.

To illustrate the generalization of our method we first solve the Pauli equation mentioned above. Then, a more general Pauli-like equation is solved, and finally, we find the spectrum of the Dirac hydrogen atom by considering the Feynman-Gell-Man equation.

2. *An Electron with Intrinsic Electric Dipole Moment*

The first-order Pauli equation for an electron having an electric dipole moment moving in a Coulomb field has been treated by Feinberg (1958) and Salpeter (1958) in connection with the upper limits of the dipole moment of the electron. The equation considered by these authors is

$$\left(\frac{p^2}{2m} - \frac{e^2}{r} - \frac{f}{2m}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r^3} - E\right)\psi = 0 \tag{2.1}$$

from where we obtain the function $D(r)$ to be introduced in (1.3). For the left side of equation (1.3) we choose $h(r) = r$, yielding

$$(L^2 - Q)\frac{1}{r^2} + \frac{2d}{r} - 4b \tag{2.2}$$

while the right-hand side of this equation is

$$f\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r}\frac{1}{r^2} + \frac{2me^2}{r} + 2mE \tag{2.3}$$

The comparison of the last two expressions gives us the following results

$$b = -\frac{mE}{2}$$

$$d = me^2$$

$$Q = \mathbf{L}^2 - f\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r} \tag{2.4}$$

These are the results that we need to find the energy spectrum by means of equation (1.5), except that we have to diagonalize the operator Q. To diagonalize Q we define the operator (Salamó, 1972)

$$G = -f\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r} - \boldsymbol{\sigma}\cdot\mathbf{L} - 1 \tag{2.5}$$

which commutes with the Casimir operator Q, the total angular momentum J^2 and the compact generator T_0 of the SGA. Therefore G can be diagonalized simultaneously with these operators. It is straightforward to show that

$$G^2 = J^2 + \frac{1}{4} + f^2$$

implying that the spectrum of G is

$$g = \pm\sqrt{(j+\tfrac{1}{2})^2 + f^2} \tag{2.6}$$

The Casimir Q can now be expressed in terms of the diagonal operators

$$Q = J^2 + \tfrac{1}{4} + G \tag{2.7}$$

Hence, its spectrum is

$$q = (j+\tfrac{1}{2})^2 + g \tag{2.8}$$

Replacing the values of b and d given in (2.4) and the value of q, (2.8), we get

$$E_n = -\frac{me^4}{2[n + \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 + g + \frac{1}{4}}]^2} \tag{2.9}$$

which coincides with the spectrum given by Feinberg (1958) in his equation (18).

A more general Pauli-like equation that we may consider with equal ease by means of our techniques is defined by the potential

$$V(r) = -\frac{e^2}{r} - \frac{1}{2m}\left(\frac{w}{r^2} + f\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r^3} + k\frac{\boldsymbol{\sigma}\cdot\mathbf{L}}{r^2}\right) \tag{2.10}$$

The values of b and d remain the same as in the previous case since (2.2) does not change and in (2.3) the only change happens in the term that behaves like r^{-2}. The difference then is in the Casimir operator which now is given by

$$Q = J^2 + \tfrac{1}{4} + (1+k)G + (k-w) \tag{2.11}$$

where this time

$$G = -\frac{f}{k+1}\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r} - \boldsymbol{\sigma}\cdot\mathbf{L} - 1$$

and its spectrum is

$$g = \sqrt{(j+\tfrac{1}{2})^2 + \left(\frac{f}{k+1}\right)^2}$$

The energy spectrum that we obtain this time is

$$E_n = -\frac{me^4}{2[n + \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 + (1+k)g + \frac{1}{4} + (k-w)}]^2} \tag{2.12}$$

If we wanted to solve a harmonic oscillator potential with the same extra σ-terms we have to choose $h(r) = r^2$, and if the dominant potential is a Morse potential we choose $h = \exp(-a(r - r_0))$. There is no difficulty in solving these cases and innumerable velocity-dependent potentials as well.

3. *The Dirac–Feynman–Gell-Mann Equation*

Finally, we derive the discrete energy spectrum of the Dirac hydrogen atom. Feynman and Gell-Mann (1958) have shown that instead of solving the four-component Dirac equation one can deal equivalently with a two component equation which, in the case of the Coulomb potential, is

$$\left(\mathbf{p}^2 - \left(E + \frac{e^2}{r}\right)^2 + ie^2\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r^3} + m^2\right)\psi = 0 \tag{3.1}$$

The situation is very similar to that of the previous section and again we choose $h(r) = r$ but now d depends on the energy and b depends quadratically on it. Proceeding as before we get

$$d = e^2E$$

$$b = \frac{m^2 - E^2}{4}$$

$$Q = \mathbf{L}^2 + ie\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r} - e^4 \tag{3.2}$$

The steps to follow now are identical to those of section 2. Defining

$$G = ie\frac{\boldsymbol{\sigma}\cdot\mathbf{x}}{r} - \boldsymbol{\sigma}\cdot\mathbf{L} - 1 \tag{3.3}$$

it results that

$$Q = G(G+1) \tag{3.4}$$

The operator G is again easily diagonalized and therefore the eigenvalues of Q to be replaced in (1.5) are directly obtained. The equation for the energy spectrum is shown to be

$$n + 1 + g = \frac{e^2 E}{\sqrt{m^2 - E^2}} \tag{3.5}$$

where $n = 0, 1, 2, \ldots$ and

$$g = \sqrt{(j + \tfrac{1}{2})^2 - e^4} \tag{3.6}$$

The energy spectrum, of course, coincides with that of Dirac.

The simplicity with which the solutions have been obtained illustrates the value and interest of the algebraic techniques.

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